Pre-Calculus Final (1122)

- Any device with a computer algebra system is prohibited during the exam.
- There are two sections in this exam. Please read the respective instruction very carefully.
- We use $\log(x)$ to denote $\log_{10}(x)$ (logarithm of base 10).

Section A. (75%)

Instruction : For each question in this section, there is only ONE correct answer. Each correct answer is worth 5%.

1. Let u, v be angles in the first quadrant, and suppose $\sin(u) = \frac{1}{3}$ and $\cos(v) = \frac{1}{4}$. Find $\sin(u+v)$.

   (A) $\frac{1+2\sqrt{30}}{12}$

   (B) $\frac{1}{12}$

   (C) $\frac{2\sqrt{2}+\sqrt{15}}{12}$

   (D) $\frac{7}{12}$

   (E) None of the above.

2. Simplify the expression
$$\sin^4 x - \cos^4 x - \sin^2 x + \cos^2 x.$$

   (A) $\sin(2x)$

   (B) $\cos(2x)$

   (C) 1

   (D) 0

   (E) None of the above.

3. Which of the following equations correctly describes a relation between the lengths p, q, r and the angles A, B, C of the following triangle ?

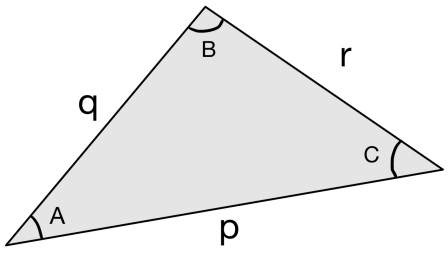
   (A) $\frac{\sin A}{p} = \frac{\sin B}{q} = \frac{\sin C}{r}$.

   (B) $\frac{\sin B}{r} = \frac{\sin C}{p} = \frac{\sin A}{q}$.

   (C) $\cos(B) = \frac{p^2+q^2-r^2}{2pq}$.

   (D) $\cos(C) = \frac{p^2+r^2-q^2}{2pr}$.

   (E) None of the above.



4. Suppose $0 \le x < 2\pi$. How many solutions does the equation $7\sin x + 2\cos^2 x = 5$ have ?

   (A) 0

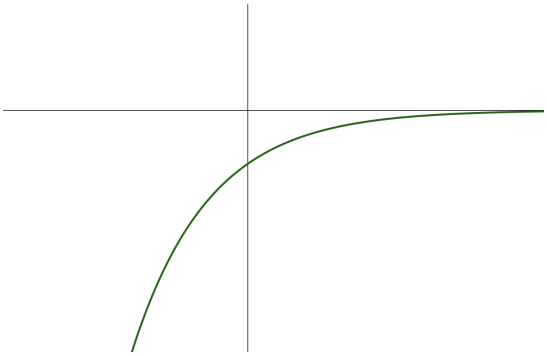
   (B) 1

   (C) 2

   (D) 3

   (E) 4

5. The following is the graph of a function $h(x)$. Which of the following can $h(x)$ possibly be?



(A) $h(x) = 2^{-x}$.
(B) $h(x) = -2^{x}$.
(C) $h(x) = -2^{-x}$.
(D) $h(x) = 2^{x} - 1$.
(E) None of the above.

6. Find x such that $5^{7x-5} = 10^{6x-7}$.

(A) $x = \dfrac{5}{9}$.
(B) $x = \dfrac{9}{5}$.
(C) $x = 7 + 5\log(5)$.
(D) $x = \dfrac{5\log(5) - 7}{7\log(5) - 6}$.
(E) None of the above.

7. For $x > 1$, rewrite the expression

$$2\log(x) - 3\log(x^2 + 1) + 4\log(x - 1)$$

as $\log(f(x))$. Find $f(x)$.

(A) $\dfrac{2x(4x-4)}{3(x^2+1)}$
(B) $x^2 - (x^2+1)^3 + (x-1)^4$
(C) $\dfrac{x^2(x-1)^4}{(x^2+1)^3}$
(D) $\dfrac{2^x \cdot 4^{x-1}}{3^{x^2+1}}$
(E) None of the above.

8. For $x, y > 0$, which of the following must be correct?

(A) $3\log(\sqrt[3]{xy}) = \log(x) + \log(y)$.
(B) $(\log(x))^y = y\log(x)$.
(C) $\log(3x^y) = \log(3) \cdot (y \log x)$.
(D) $\log(4) = (\log(2))^2$.
(E) None of the above.

9. It is known that $4x + y = 10$ and $10^{-2x} = 6 \cdot 10^{-3y}$. Which of the following must be correct?

(A) $x = \dfrac{10 + 2\log 6}{7}$.

(B) $y = \dfrac{10 + 2\log 6}{7}$.

(C) $x = \dfrac{30 + \log 6}{14}$.

(D) $y = \dfrac{30 - \log 6}{14}$.

(E) None of the above.

10. Let $A = (2, -5, -4)$, $B = (4, 1, -4)$ and $P = (k, k, k)$. It is known that $\overrightarrow{AB}$ is perpendicular to $\overrightarrow{AP}$. Find the value of k.

(A) $-14/3$

(B) $-13/4$

(C) $13/4$

(D) $14/3$

(E) None of the above is correct.

11. The intersection point P of the plane $3x + y - 2z = 23$ and the line $\vec{r}(t) = \langle 0, 2, 3\rangle + t\langle -1, 0, 3\rangle$ is

(A) $(-3, 2, 9)$

(B) $(3, -2, -9)$

(C) $(-3, -2, 6)$

(D) $(3, 2, -6)$

(E) None of the above.

12. Which of the following statement must be correct?

(A) Any two distinct lines in the space $\mathbb{R}^3$ would either be parallel or intersect.

(B) The equation $2x + 3y + 4z = 5$ represents a straight line in the space $\mathbb{R}^3$.

(C) Two non-parallel planes would intersect to give a straight line in the space $\mathbb{R}^3$.

(D) Two vectors $\vec{u}$ and $\vec{v}$ are perpendicular to each other if $\vec{u} \cdot \vec{v} = -1$.

(E) None of the above is correct.

13. The line of the intersection of planes $x + 2y - z = 5$ and $x - 2y + 2z = 5$ is

(A) $\vec{r}(t) = \langle 5 + 2t,\ 1 + 3t,\ -4t\rangle$, for $t \in \mathbb{R}$.

(B) $\vec{r}(t) = \langle 5 + 2t,\ 1 - 3t,\ -4t\rangle$, for $t \in \mathbb{R}$.

(C) $\vec{r}(t) = \langle 1 + 2t,\ 6 + 3t,\ 8 - 4t\rangle$, for $t \in \mathbb{R}$.

(D) $\vec{r}(t) = \langle 1 + 2t,\ 6 - 3t,\ 8 - 4t\rangle$, for $t \in \mathbb{R}$.

(E) None of the above.

14. Choose the correct statement.

(A) $\displaystyle\sum_{n=11}^{20} 3^n = \frac{3^{21} - 3^{11}}{2}$.

(B) $\displaystyle\sum_{n=11}^{20} n^2 = 2485$.

(C) $\displaystyle\sum_{n=11}^{20} 2n = 310$.

(D) All of the above.

(E) None of the above.

15. Which of the following is the sum $\frac{1}{n}\sum_{k=1}^{n}\left(2+3\cdot\frac{k}{n}\right)^2$ equal to?

(A) $4+6\cdot\frac{1+n}{n}+\frac{3(n+1)(2n+1)}{2n^2}$

(B) $4+12\cdot\frac{1+n}{n}+\frac{9(n+1)(2n+1)}{2n^2}$

(C) $4+12\cdot\frac{1+n}{n}+\frac{3(n+1)(2n+1)}{2n^2}$

(D) $4+6\cdot\frac{1+n}{n}+\frac{9(n+1)(2n+1)}{2n^2}$

(E) None of the above.

Section B. (25%)

Instruction : For each question in this section, choose all correct answers. For each question,

- if you make no mistakes in your choices, you get 5%;
- if you make one mistake in your choices, you get 4%;
- if you make two mistakes in your choices, you get 3%;
- if you make more than two mistakes in your choices, you get 0%.

16. Determine which of the following are solutions to the equation

$$4\sin^3 x - 8\sin^2 x - \sin x + 2 = 0.$$

(A) $\frac{\pi}{6}$

(B) $\frac{3\pi}{4}$

(C) $\frac{5\pi}{6}$

(D) $\frac{7\pi}{4}$

(E) $\frac{11\pi}{6}$

17. Which of the following expressions are correct?

(A) $\log_3(3^{-7x+3}) = -7x+3$ for any real numbers x.

(B) $4^{\log_4(-9+6q)} = -9+6q$ for any $q > \frac{3}{2}$.

(C) $7^{6\log_7(9)-9\log_7(6)} = 2^{-9}\cdot 3^3$.

(D) The domain of $f(x) = \log_8(5-5x)$ is $x \le 1$.

(E) The range of $f(x) = \log_8(5-5x)$ is $\mathbb{R}$ (the set of all real numbers).

18. Consider three points $A = (1,1,1)$, $B = (2,3,0)$, $C = (0,1,3)$. Let P be the plane containing these three points. Which of the following statements must be correct?

(A) $\langle -4,-1,-2\rangle$ is a normal vector of the plane P.

(B) The area of $\triangle ABC$ is $\frac{\sqrt{21}}{2}$.

(C) The line $\vec{r}(t) = \langle 1+t, 1, 1-2t\rangle$ lies on the plane P.

(D) The angle $\angle BAC$ is greater than $\frac{\pi}{2}$.

(E) The equation of the plane P is $4x - y + 2z = 0$.

19. Suppose that $\|\vec{v}\| = 1$, $\|\vec{w}\| = 3$ and the angle between $\vec{v}$, $\vec{w}$ is $\dfrac{\pi}{3}$. Which of the following statements must be correct?

(A) $\|\vec{v} + 2\vec{w}\| = \sqrt{43}$.

(B) The angle between $\vec{v} + 2\vec{w}$ and $\vec{v}$ is $\dfrac{\pi}{3}$.

(C) $\|(\vec{v} + 2\vec{w}) \times \vec{v}\| = 3\sqrt{3}$.

(D) $(\vec{v} + 2\vec{w}) \bullet (\vec{v} - 2\vec{w}) = 7$.

(E) $(\vec{v} + 2\vec{w}) \times \vec{v}$ is perpendicular to $\vec{v}$ and $\vec{w}$.

20. Suppose $\{a_k\}_{k=1}^{\infty}$ is a sequence such that $\sum_{k=1}^{n} a_k = 10n^3 + 4n$. Which of the following statements must be correct?

(A) $a_1 = 14$.

(B) $a_n = 30n^2 - 30n + 14$.

(C) $\sum_{k=1}^{n-2} a_{k+2} = 10n^3 + 4n - 88$.

(D) We cannot determine the exact formula of a_n from the given information.

(E) $\{a_k\}_{k=1}^{\infty}$ is an arithmetic sequence.